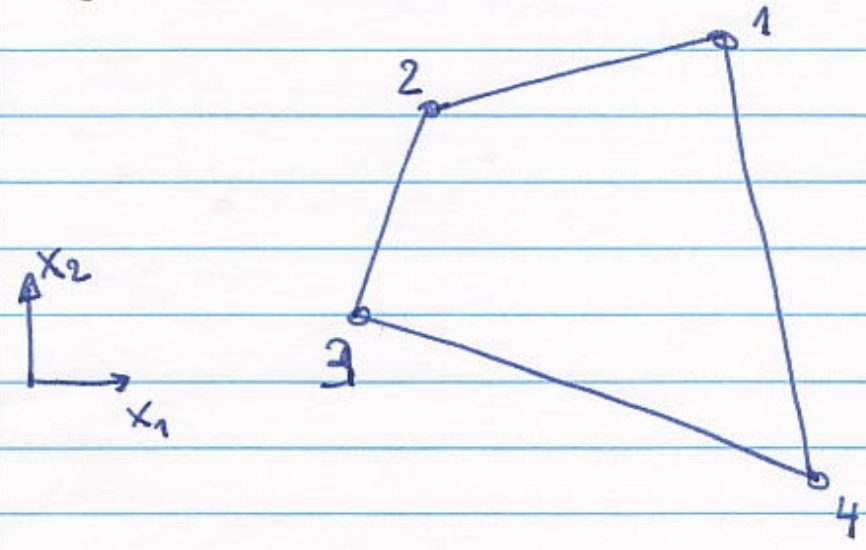


Formulation of isoparametric elements (Bathe's book)

Consider the quadrilateral element shown in the figure:



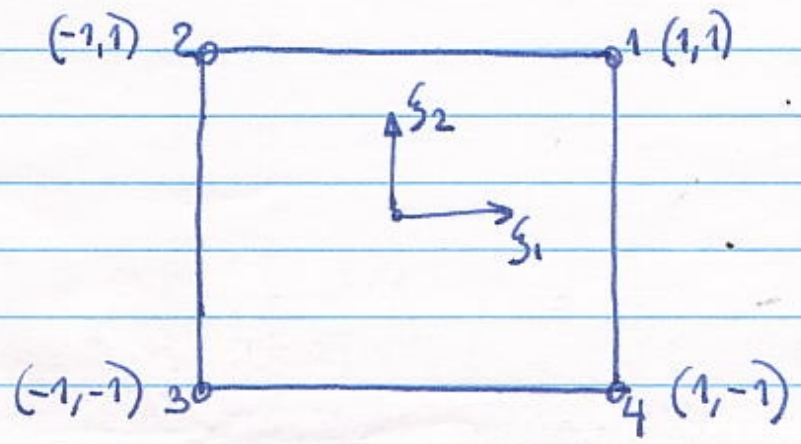
nodal coordinates:

$$X_i^a, i=1,2, a=1,-1,4$$

nodal displacements

$$U_i^a$$

We need to generalize our interpolation for the linear square. Consider the mapping from the following master element



We interpolate the displacement field as before

$$u = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [H] \{U\}, \text{ where } U \text{ are the nodal displacements}$$

$$H = \begin{bmatrix} \frac{1}{2} (1+\xi_1)(1+\xi_2) & 0 & \dots & \dots \\ 0 & \frac{1}{2} (1+\xi_1)(1+\xi_2) & \dots & \dots \end{bmatrix}$$

So $u = u(\xi_1, \xi_2)$

But we need expression for $\frac{\partial u_1}{\partial x_1}, \dots$, (i.e. strains).

We follow the following procedure:

- Define the mapping (interpolation) from the master element to the quadrilateral:

$$x(\xi) = \begin{Bmatrix} x_1(\xi_1, \xi_2) \\ x_2(\xi_1, \xi_2) \end{Bmatrix} = [H] \{X\}$$

where $\{X\}$ is the vector of nodal coordinates

$$\{X\}^T = \{X_1^1 \ X_2^1 \ X_1^2 \ X_2^2 \ \dots \ X_1^4 \ X_2^4\}$$

• Link derivatives through chain rule

$$\frac{\partial f}{\partial \xi_1} = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial \xi_1} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial \xi_1}$$

$$\frac{\partial f}{\partial \xi_2} = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial \xi_2} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial \xi_2}$$

In vector form:

$$\begin{Bmatrix} \frac{\partial f}{\partial \xi_1} \\ \frac{\partial f}{\partial \xi_2} \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_1} \\ \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_2} \end{bmatrix}}_J \begin{Bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{Bmatrix}$$

What we really need is the inverse:

$$\begin{Bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{Bmatrix} = J^{-1} \begin{Bmatrix} \frac{\partial f}{\partial \xi_1} \\ \frac{\partial f}{\partial \xi_2} \end{Bmatrix}, \quad J^{-1} = \frac{1}{\det J} \begin{bmatrix} \frac{\partial x_2}{\partial \xi_2} & -\frac{\partial x_2}{\partial \xi_1} \\ -\frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_1} \end{bmatrix}$$

• Now we can derive the interpolation of strains

$$\epsilon = \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \begin{Bmatrix} \partial u_1 / \partial x_1 \\ \partial u_2 / \partial x_2 \\ \partial u_1 / \partial x_2 + \partial u_2 / \partial x_1 \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} \end{Bmatrix} = J^{-1} \begin{Bmatrix} \frac{\partial u_1}{\partial \xi_1} \\ \frac{\partial u_1}{\partial \xi_2} \end{Bmatrix}, \text{ same for } u_2$$

The strain vector can then be written as:

$$\{ \epsilon \} = [A] \begin{Bmatrix} \partial u_1 / \partial \xi_1 \\ \partial u_1 / \partial \xi_2 \\ \partial u_2 / \partial \xi_1 \\ \partial u_2 / \partial \xi_2 \end{Bmatrix}, \text{ where}$$

3×4 4×1

$$[A] = \frac{1}{\det J} \begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \text{ and}$$

$$= [G] \{ U \}, [G] \text{ is the matrix of derivatives of our shape functions}$$

4×8 8×1

$$\Rightarrow \begin{matrix} 3 \times 1 & 3 \times 4 & 4 \times 8 & 8 \times 1 \\ \{\varepsilon\} = & \underbrace{[A][G]}_B & \{U\} \end{matrix}$$

3×8

- The final step is the computation of the stiffness matrix for the element:

$$[K^e] = \int_{\Omega^e} B^T C B \, dV$$

$$[K^e] = \int_{-1}^1 \int_{-1}^1 B^T C B J \, d\xi_1 \, d\xi_2$$

- Element force vector

$$[R^e] = \int_{\Omega^e} [H]^T \{f\} \, dV + \int_{S^e} [H]^T \{t\} \, dS$$

~~$$[R^e] = \int_{\Omega^e} [H]^T \{f\} \, dV + \int_{S^e} [H]^T \{t\} \, dS$$~~