

Unit #12
The Ritz Method (cont'd)

$$\frac{d}{dx}(EAu') = f(x) \quad 0 < x < L \quad (\text{Model problem})$$

$$\text{Potential: } \Pi = \frac{1}{2} \int_0^L AE u'^2 dx - \int_0^L f u dx$$

Boundary conditions: 

$$\begin{array}{ll} u(0) = 0 & EAu'(L) = P \\ (\text{essential}) & (\text{natural}) \end{array}$$

Ritz Approximation: $u \sim c_i \phi_i(x) \quad i=1, N$

$$\Pi(u) \sim \Pi(c_i \phi_i) = \Pi(c_i)$$

$$\text{Equilibrium: } \delta \Pi = \frac{\partial \Pi}{\partial c_i} \delta c_i = 0$$

$\Rightarrow N \times N$ system of equations to determine c_i

$$\Pi = \frac{1}{2} \int_0^L EA (c_i \phi_i')^2 dx - \int_0^L f c_i \phi_i dx$$

$$\frac{\partial \Pi}{\partial c_i} = \underbrace{\int_0^L EA \phi_i' \phi_j' dx}_{K_{ij}} c_i - \underbrace{\int_0^L f \phi_i dx}_{R_i}$$

Solve for c_i : $[K]\{c\} = \{R\}$

$$\{c\} = [K]^{-1}\{R\}$$

Reconstruct approximate solution from obtained coefficients.

Alternative formulation using PVD:

$$\text{PVD: } \int_0^L \sigma \delta \epsilon A dx = \int_0^L f \delta u dx \quad \forall \text{ admissible } \delta u$$

$$\sigma = E u', \quad \text{approximate } u \sim c_i \phi_i, \quad \text{as usual}$$

Approximate the virtual displacements as:

$$\delta u = \delta c_i \phi_i, \quad \text{then}$$

$$\delta \epsilon = \delta c_i \phi_i'$$

$$\int_0^L EA C_i \phi_i' \phi_j' dx \delta C_j = \int_0^L f \phi_k dx \delta C_k$$

$$\text{or } \left[\underbrace{\int_0^L EA \phi_j' \phi_i' dx}_{K_{ij}} C_j - \underbrace{\int_0^L f \phi_i dx}_{R_i} \right] \delta C_i = 0 \quad \forall \delta C_i$$

$$\text{or } [K] \{C\} = \{R\} \quad \text{as before.}$$

Conditions on ϕ_i 's

- ① • Want approximate solutions that become closer to exact solution "u" as "N" is increased
- ② • Want ϕ_i such that conditions of PVD are satisfied.
- ③ • Want system $[K] \{C\} = \{R\}$ to have unique solution (linearly independent equations).
- ϕ_i some continuity such that integrals in K_{ij} can be evaluated (exist, $< \infty$)
- satisfy the essential boundary conditions

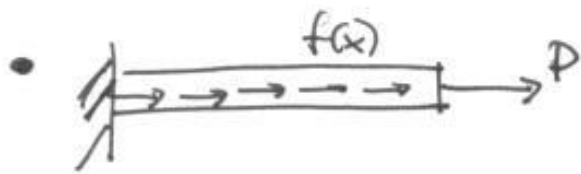
if $u = \bar{u} \neq 0$ on some part of S_u
 we satisfy this by requiring:

$\phi_1(x) = \bar{u}$ and all others = 0 on this part of
 the boundary.

- the set of functions $\{\phi_i\}$ must be "complete"

These conditions do not provide guidelines for
 generating the functions.

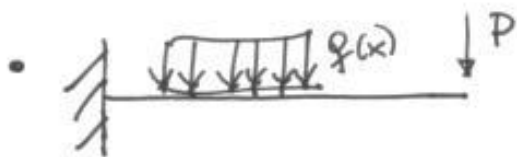
Usually one adopts a family of simple functions
 (polynomials, trigonometric functions) satisfying the
 requirements above.



• $\phi_i = x^i$

• $\phi_i = \sin\left[\frac{(2i+1)\pi x}{2L}\right]$

note that $\phi_i = \sin\frac{\pi i x}{L}$ would give $U(L) = 0$

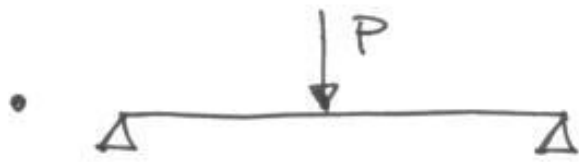


$w_0 \sim C_i \phi_i$, $\phi_i = \sin\left[\frac{(2i+1)\pi x}{2L}\right]$

$\phi_i = x^i$



Convergence could be very slow for a poor choice of basis functions ϕ_i :



solution is two piecewise cubic polynomials.

$$\phi_i = \sin \frac{\pi i x}{L} \quad \text{gives very slow convergence.}$$

Remarks:

- for well-chosen ϕ_i 's the process converges (proof omitted)
- for increasing "N" the previously computed c_i 's don't change.
- K_{ij} is symmetric for linear elasticity
- strains and stresses are generally
- governing equation and natural boundary condition satisfied in the variational (integral) sense. Therefore the equation of equilibrium is not satisfied pointwise.
- ~~since a continuous system (∞ degrees of freedom) is approximated with a finite number~~
- from PMPE, approximate solution minimizes energy within subspace of functions \Rightarrow not the real minimum \Rightarrow energy is higher \Rightarrow system is stiffer.