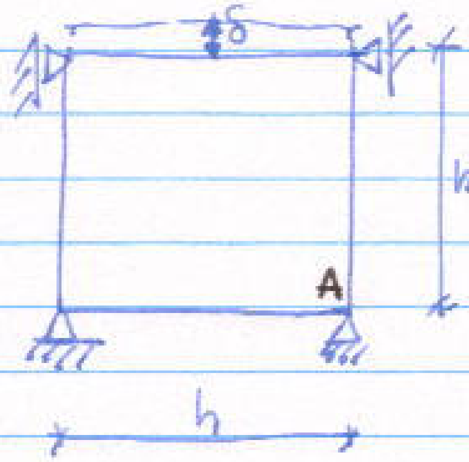
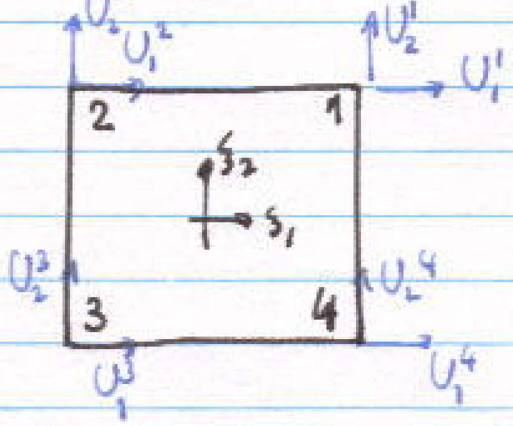
Practice Problems

(1)





$$[K^e]\{U^e\} = \{R^e\} \qquad \{U^e\}^T = \{0\ \delta\ 0\ \delta\ 0\ 0\ 0\ 0\}$$

The reaction at A (horizontal) is obtained from the equation:

$$K_{71} \times 0 + K_{72}\,\delta + 0 + K_{74}\,\delta + 0 \ldots = R_1^4 = H_A$$

$$\frac{E\,(1-4\nu)}{8\,(2\nu^2+\nu-1)}\,\delta + \frac{E}{8\,(1+\nu+2\nu^2)}\,\delta = R_1^4 = H_A$$

$$\boxed{H_A = \frac{\nu E\,\delta}{2\,(1-\nu-2\nu^2)}}$$

Exact: $\varepsilon_{22} = \dfrac{\delta}{h}$, $\varepsilon_{11} = 0 = \dfrac{\sigma_{11}}{E} - \dfrac{\nu}{E}(\sigma_{22}+\sigma_{33})$

$\varepsilon_{33} = 0 = \dfrac{\sigma_{33}}{E} - \dfrac{\nu}{E}(\sigma_{11}+\sigma_{22})$

$$\begin{cases} \sigma_{11} = \nu\left(\sigma_{22} + \sigma_{33}\right) \\ \sigma_{33} = \nu\left(\sigma_{22} + \sigma_{11}\right) \end{cases}$$

$$\sigma_{11} + \sigma_{33} = 2\nu\sigma_{22} + \nu\left(\sigma_{11} + \sigma_{33}\right), \quad \sigma_{11} - \sigma_{33} = \nu\left(\sigma_{33} - \sigma_{11}\right)$$

$$\sigma_{11} + \sigma_{33} = \frac{2\nu\sigma_{22}}{1-\nu} \qquad \sigma_{11}\cancel{(1+\nu)} = \sigma_{33}\cancel{(1+\nu)}$$

$$E\varepsilon_{22} = \frac{E\delta}{h} = \sigma_{22} - \nu\,\frac{2\nu\sigma_{22}}{1-\nu} = \sigma_{22}\,\frac{1-\nu-2\nu^2}{1-\nu}$$

$$\sigma_{22} = \frac{E\delta}{h}\,\frac{(1-\nu)}{(1+\nu)(1-2\nu)}$$

$$\cancel{2}\sigma_{11} = \frac{\cancel{2}\nu}{\cancel{(1-\nu)}}\,\frac{E\delta\,\cancel{(1-\nu)}}{h(1+\nu)(1-2\nu)}$$

$$\sigma_{11} = \frac{\nu E\delta}{h(1+\nu)(1-2\nu)}$$

The reaction H_A is half the total load:

$$H_A = \frac{\nu E\delta\,\cancel{h}}{2\cdot\cancel{h}}\,\frac{1}{(1+\nu)(1-2\nu)}$$

$$H_A = \frac{\nu E\delta}{2}\,\frac{1}{(1+\nu)(1-2\nu)} \qquad \text{SAME!!}$$

2 $\left(EI\,w_0''\right)'' + \left(N\,w_0'\right)' = 0 \qquad 0<x<L$

Multiply by suitable virtual deflection field δw_0 and integrate between "0" and "L":

$$\int_0^L \left(EI\,w_0''\right)''\delta w_0\,dx + \int_0^L \left(N\,w_0'\right)'\delta w_0\,dx = \int_0^L 0\,\delta w\,dx$$

Integrate by parts:

$$\int_0^L \Big[\big(\underbrace{EI\,w_0''}_{M}\big)'\delta w_0\Big]'dx - \int_0^L \left(EI\,w_0''\right)'\delta w_0'\,dx + \int_0^L \left(N w_0'\,\delta w_0\right)'dx$$

$$-\int_0^L N\,\delta w_0'\,w_0'\,dx = 0$$

$$\big(\underbrace{EI\,w_0''}_{M}\big)'\delta w_0\Big|_0^L - \int_0^L \left(M\,\delta w_0'\right)'dx + \int_0^L M\,\delta w_0''\,dx +$$

$$+\;N\,w_0'\,\delta w_0\Big|_0^L - \int_0^L N\,w_0'\,\delta w_0'\,dx = 0$$

$$M'\,\delta w_0\Big|_0^L + N\,w_0'\,\delta w_0\Big|_0^L - M\,\delta w_0'\Big|_0^L + \int_0^L M\,\delta w_0''\,dx -$$

$$-\int_0^L N\,w_0'\,\delta w_0'\,dx = 0$$

The PVD reads:

$$\int_0^L M\,\delta w_0''\,dx = \int_0^L N\,w_0'\,\delta w_0'\,dx \qquad 0<x<L$$

$\forall\ \delta w_0$ s.t. $\delta w_0, \delta w_0'$ satisfy the homogeneous essential boundary conditions

The natural boundary conditions are not enforced by the principle.

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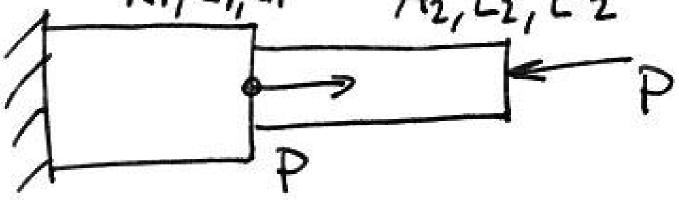
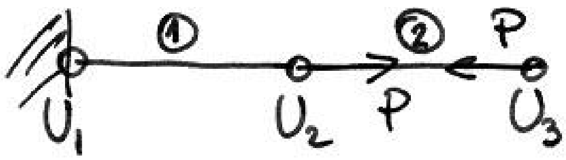


By equilibrium reaction at built-in end is "0", as is the normal force in the left member.

The exact solution is:

$$\sigma_1 = 0 \qquad \sigma_2 = \frac{P}{A_2}$$

Finite element solution:



$$K^e = \frac{A^e E^e}{L^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}}_{K} \begin{Bmatrix} 0 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} P \\ -P \end{Bmatrix}$$

$$\begin{cases} K_{22}\, U_2 + K_{23}\, U_3 = P \\ K_{32}\, U_2 + K_{33}\, U_3 = -P \end{cases}$$

$$K_{22} = K_{22}^{(1)} + K_{11}^{(2)} = \frac{A_1 E_1}{L_1} + \frac{A_2 E_2}{L_2}$$

$$K_{23} = K_{12}^{(2)} = \frac{-A_2 E_2}{L_2} = K_{32}$$

$$K_{33} = K_{22}^{(2)} = \frac{A_2 E_2}{L_2}$$

Solving system:

$$(K_{22} + K_{32}) U_2 + 0\, U_3 = 0 \Rightarrow \boxed{U_2 = 0}$$

$$K_{33} U_3 = -P$$

$$\rightarrow \boxed{U_3 = \frac{-P L_2}{A_2 E_2}}$$

Element 1 $\quad U_1^{(1)} = U_1 = 0 , \quad U_2^{(1)} = U_2 = 0$

$$u^{(1)} = \phi_1 0 + \phi_2 0 = 0$$

$$\varepsilon^{(1)} = 0 , \quad \sigma^{(1)} = 0$$

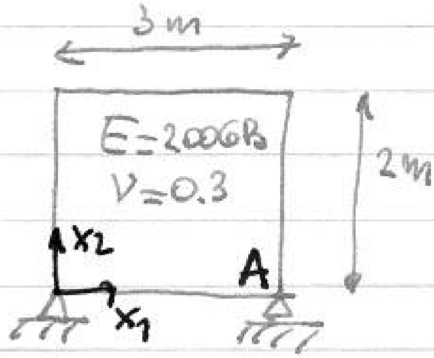
Element 2 $\quad U_1^{(2)} = U_2 = 0 \quad U_2^{(2)} = U_3 = \frac{-P L_2}{E_2 A_2}$

$$u^{(2)} = \phi_1 0 + \phi_2 \left(\frac{-P L_2}{E_2 A_2} \right) , \quad \phi_2 = \frac{x}{L_2}$$

3c

$$\varepsilon^{(2)} = \frac{-P}{E_2 A_2} \quad , \quad \sigma^{(2)} = \frac{-P}{A_2} \quad \text{Exact!}$$

4) The 4-node plane strain element shown is subjected to the constant stresses



$\sigma_{11} = 200$ MPa
$\sigma_{22} = 100$ MPa (loads not shown)
$\sigma_{12} = 100$ MPa

Compute the displacement at A

$$\varepsilon_{22} = 0 \longrightarrow \sigma_{33} = \nu(\sigma_{11} + \sigma_{22}) = 90 \text{ MPa}$$

$$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})\right] = \frac{143\ 10^6}{200\ 10^9}$$

$$u = X_1 \varepsilon_{11} + X_2 2\varepsilon_{12}$$

$$2\varepsilon_{12} = \frac{\sigma_{12}}{G} = \frac{100 \text{ MPa}}{200 \text{ GPa}}\, 2(1 + 0.3) = \frac{2.6}{2\ 10^3}$$

$$u(3,0) = 3\ \varepsilon_{11} = \frac{3 \times 143}{200\ 10^3} \text{ m} = 0.21 \text{ cm}$$

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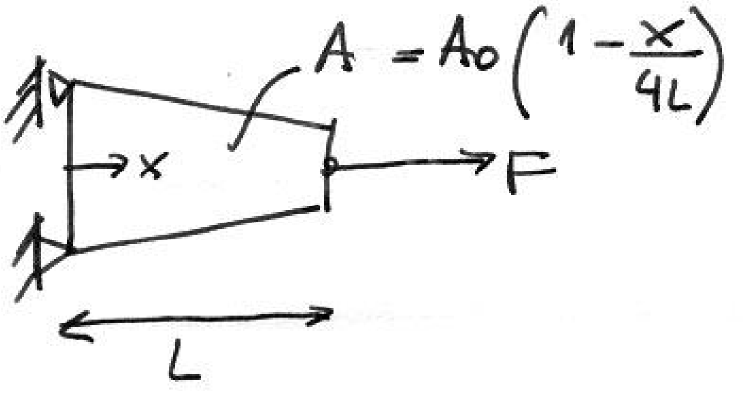

PVD: $\int_V \sigma_{ij}\, \delta\varepsilon_{ij}\, dV = F\, \delta u(L)$

specializing to 1D: $\int_0^L A(x)\sigma\, \delta\varepsilon\, dx = F\, \delta u(L)$

$$\int_0^L \cancel{A_0}\left(1-\frac{x}{4L}\right)\left(\frac{72}{73}+\frac{24x}{73L}\right)\frac{\cancel{F}}{\cancel{A_0}}\,\delta\varepsilon\, dx \overset{?}{=} \cancel{F}\,\delta u(L)$$

1 $\delta u = \dfrac{a x}{L}$ (admissible, satisfies homo. essential B.C. $\delta u(0)=0$).

$$\delta\varepsilon = \frac{a}{L}$$

PVD: $\displaystyle\int_0^L \left(1-\frac{x}{4L}\right)\left(\frac{72}{73}+\frac{24}{73}\frac{x}{L}\right)\frac{\cancel{a}}{L}\, dx \overset{?}{=} \cancel{a}\, 1$

$$\frac{72}{73}-\frac{1}{4}\frac{72}{73}\frac{1}{2}+\frac{24}{73}\frac{1}{2}-\frac{24}{4\times 73}\frac{1}{3} \overset{?}{=} 1$$

$$\frac{7}{8}\frac{72}{73}+\frac{24}{73}\left(\frac{1}{2}-\frac{1}{12}\right)\stackrel{?}{=}1$$

$$\left(\frac{21}{8}+\frac{5}{12}\right)\frac{24}{73}\stackrel{?}{=}1$$

$$\cancel{\frac{63+10}{24}}\,\cancel{\frac{24}{73}}\stackrel{?}{=}1 \quad \checkmark$$

② $\delta u = a\, x^3/L^3 \;,\; \delta\varepsilon = 3\,a\,x^2/L^3$

$$\int_0^L\left(1-\frac{x}{4L}\right)\left(\frac{72}{73}+\frac{24}{73}\frac{x}{L}\right)\frac{\cancel{3a}\,x^2}{L^3}\,dx\stackrel{?}{=}\cancel{a}\,1\times\frac{\cancel{3}}{3}$$

$$\frac{72}{73}\frac{1}{3}-\frac{1}{4}\frac{72}{73}\frac{1}{4}+\frac{24}{73}\frac{1}{4}-\frac{1}{4}\frac{24}{73}\frac{1}{5}\stackrel{?}{=}\frac{1}{3}$$

$$\frac{72}{73}\left(\frac{1}{3}-\frac{1}{16}\right)+\frac{24}{73}\left(\frac{1}{4}-\frac{1}{20}\right)\stackrel{?}{=}\frac{1}{3}$$

$$\frac{72}{73}\frac{13}{48}+\frac{24}{73}\frac{1}{5}\stackrel{?}{=}\frac{1}{3}$$

PVD not satisfied
stresses not in
equilibrium !!

$$\frac{24}{73}\left(\frac{\cancel{3}\times 13}{\cancel{48}\,16}+\frac{1}{5}\right)\stackrel{?}{=}\frac{1}{3}$$

$$\frac{24}{73}\left(\frac{65+16}{80}\right)=\frac{243}{730}=\boxed{0.3328\neq\frac{1}{3}}$$