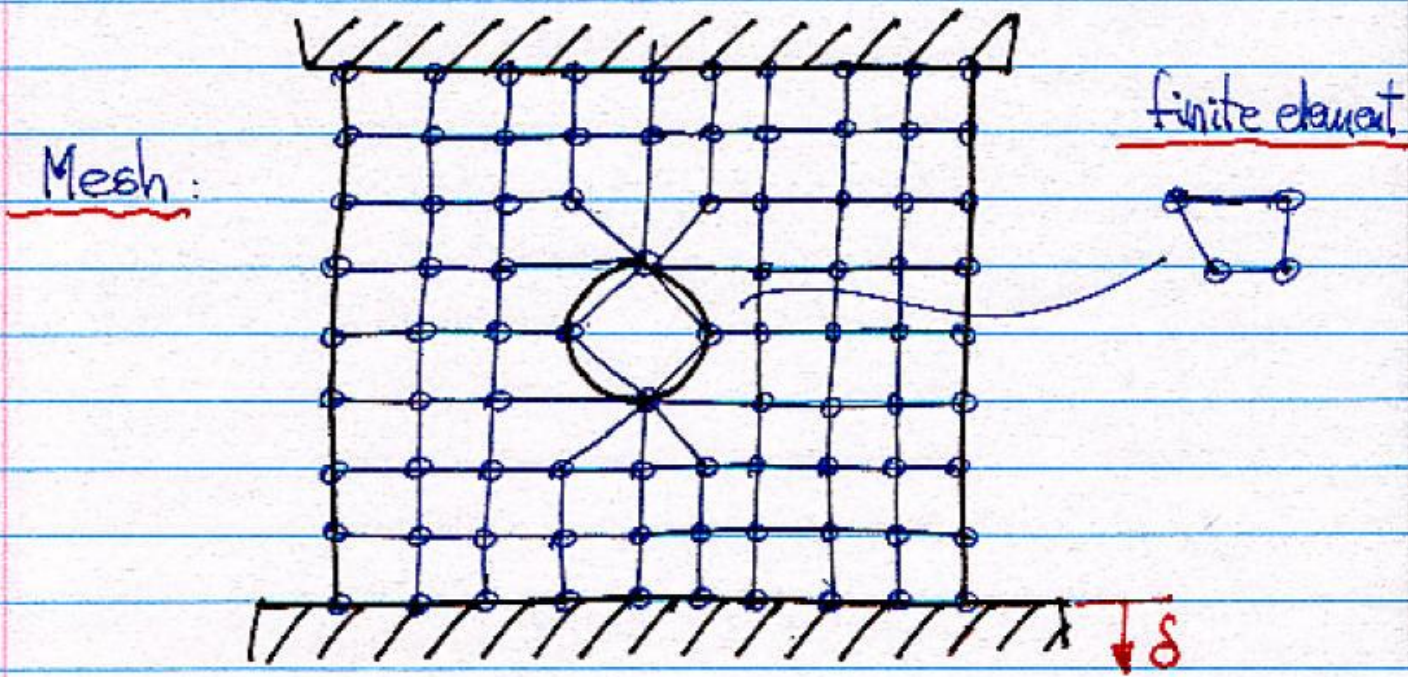
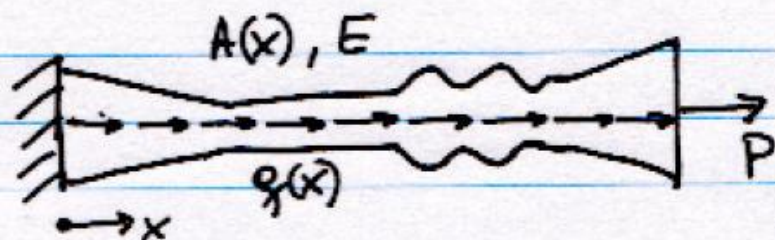


# The Finite Element Method

- Overcome limitations of Ritz:
- Simple basis functions (low order polynomials)
- Basis functions supported in subdomains (finite elements)
- Basis functions constructed to provide interpolant of approximate solution.
- Undetermined parameters represent values of dependent variables (solution) at subdomain boundaries.



Model problem:

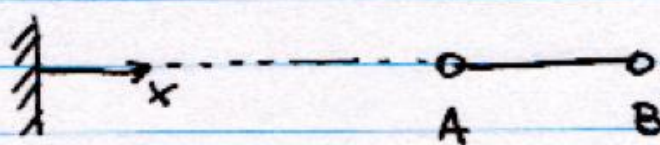


$$\frac{d}{dx} \left( EA(x) \frac{du}{dx} \right) + q(x) = 0 \quad 0 < x < L$$

BC: "u" specified:  $u(0) = 0$

"EA du/dx" specified:  $EA \frac{du}{dx} \Big|_L = P$

Formulation of generic element:



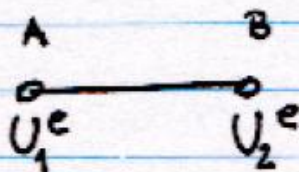
$$\Omega^e: \begin{pmatrix} x_A, x_B \\ x_1^e, x_2^e \end{pmatrix}$$

Seek variational approximation in this domain:

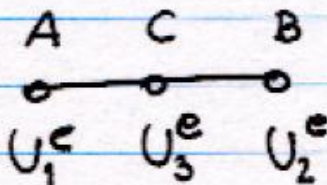
$$u(x) \approx u_e(x) = \sum_{i=1}^n \phi_i^e(x) U_i^e \quad x_A < x < x_B$$

n: number of "nodes" in element

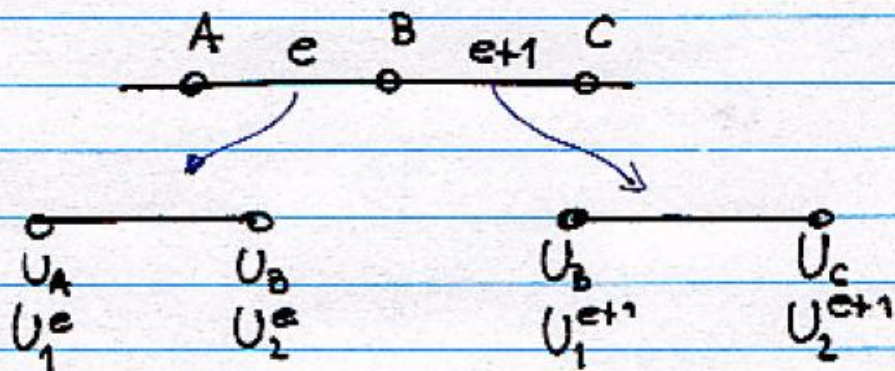
$n=2$ :



$n=3$ :



- Note: choosing undetermined parameters the values of the solution at the nodes enforces continuity of the solution across elements.



This imposes conditions on  $\phi_i^e$ :

$$\begin{aligned}
 u_e(x_j) &= \sum_{i=1}^n \phi_i^e(x_j) U_i^e = U_j^e \\
 &= \underbrace{\phi_1^e(x_j)}_0 U_1^e + \dots + \underbrace{\phi_j^e(x_j)}_1 U_j^e + \dots + \underbrace{\phi_n^e(x_j)}_0 U_n^e
 \end{aligned}$$

or  $\phi_i^e(x_j) = \delta_{ij}$

$\Rightarrow$  Lagrange polynomials

$$\phi_j^e = \prod_{\substack{k=1 \\ k \neq j}}^n \frac{x - x_k^e}{x_j^e - x_k^e}$$

$$\phi_j^e(x) = \frac{(x-x_1) \dots (x-x_{j-1}) \dots (x-x_{j+1}) \dots (x-x_n)}{(x_j-x_1) \dots (x_j-x_{j-1}) \dots (x_j-x_{j+1}) \dots (x_j-x_n)}$$

$$\phi_j^e(x_i) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Another important property that the  $\phi_i^e$  must satisfy is to allow for the representation of constant solutions exactly:

$$u(x) = C = \sum_{i=1}^n \phi_i^e(x) U_i^e \quad x_A < x < x_B$$

$$= \sum_{i=1}^n \phi_i^e(x) C = C \underbrace{\sum_{i=1}^n \phi_i^e}_1$$

•  $\sum \phi_i^e(x) = 1$

Examples:

$n=2$ :  $\phi_j^e(x) = \prod_{\substack{k=1 \\ k \neq j}}^2 \frac{x-x_k}{x_j-x_k}$

$$\phi_1^e = \frac{x - x_2}{x_1 - x_2} \quad \phi_2^e = \frac{x - x_1}{x_2 - x_1}$$

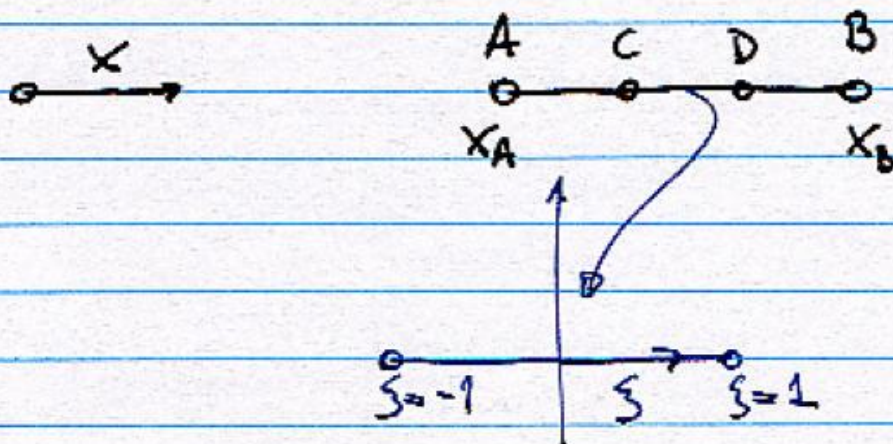
$n=3$ :

$$\phi_1^e = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} \quad \phi_2^e = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}$$

$$\phi_3^e = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

### Natural coordinate system

The functions look simpler when expressed in terms of the local coordinate system:



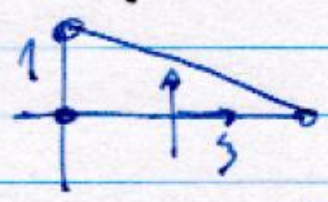
The transformation from " $\xi$ " to " $x$ " is

$$x = \frac{1 - \xi}{2} x_A + \frac{1 + \xi}{2} x_B$$

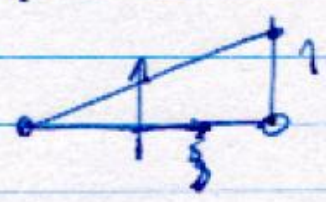
$\eta=2:$  
$$\phi_1 = \frac{x-x_B}{x_A-x_B} = \frac{1}{x_A-x_B} \left[ \frac{x_A+x_B}{2} + \frac{x_B-x_A}{2} \xi - x_B \right]$$

$$= \frac{1}{x_A-x_B} \left[ \frac{x_A-x_B}{2} - \frac{x_A-x_B}{2} \xi \right]$$

$$= \frac{1}{2} (1 - \xi)$$

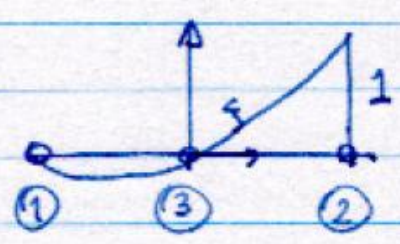
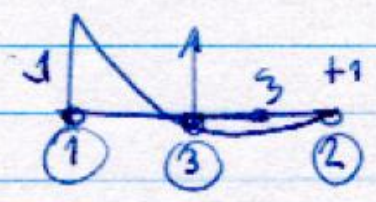


$$\phi_2 = \frac{1}{2} (1 + \xi)$$



$\eta=3:$  
$$\phi_2 = \frac{1}{2} \xi (\xi + 1)$$

$$\phi_3 = \frac{1}{2} \xi (1 - \xi)$$



$$\phi_3 = (1 - \xi)(1 + \xi)$$

