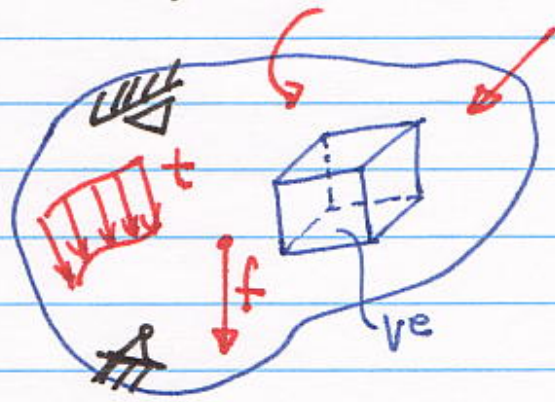


The finite element method (I)  
for three-dimensional  
elasticity problems

Potential energy applied to one element

$$\pi^e = \int_{V^e} \frac{1}{2} C_{ijkl} \epsilon_{kl} \epsilon_{ij} dV - \int_{V^e} f_i u_i dV - \int_{S^e} t_i u_i dS$$

Here is the picture:



Introduce an approximation for the displacement field "u<sub>i</sub>" within the element:

$$u_i^e = \phi_k^e U_{ik}^e$$

$$\underline{i=1,3 \quad k=1,n}$$

n: number of nodes  
per element

What can be inferred about the approximation

for  $u_1, u_2, u_3$ ?

Approximation for strains:

$$\epsilon_{ij}^e = \frac{1}{2} (u_{i,j}^e + u_{j,i}^e) \quad \text{drop "e's"}$$

$$= \frac{1}{2} (\phi_{k,j} U_{k,i} + \phi_{k,i} U_{k,j})$$

Procedure is the same as before:

- replace in potential
- minimize with respect to nodal displacements  $U_{ik}$
- $\Rightarrow$  obtain finite element matrices

stiffness  $[K]$

force vector  $[R]$

Expressions look simpler if we write in matrix form

$$\bullet \mu_i \rightarrow \mu \equiv \begin{Bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{Bmatrix} \in \mathbb{R}^{3 \times 1}$$

$$\bullet \varepsilon_{ij} \rightarrow \varepsilon \equiv \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{13} \\ \varepsilon_{23} \\ \varepsilon_{12} \end{Bmatrix} \in \mathbb{R}^{6 \times 1}$$

$$\bullet \sigma_{ij} \rightarrow \sigma \equiv \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{13} \\ \sigma_{23} \\ \sigma_{12} \end{Bmatrix} \in \mathbb{R}^{6 \times 1}$$

$$\bullet \sigma = C \varepsilon, \quad C \in \mathbb{R}^{6 \times 6}$$

$$\bullet f_i \rightarrow f \equiv \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} \in \mathbb{R}^{3 \times 1}$$

$$\bullet t_i \rightarrow t \equiv \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} \in \mathbb{R}^{3 \times 1}$$

The potential now reads:

$$\Pi^e = \int_{V^e} \frac{1}{2} \epsilon^T C \epsilon dv - \int_{V^e} u^T \cdot f dv - \int_S u^T \cdot t ds$$

$$\frac{1 \times 6 \quad 6 \times 6 \quad 6 \times 1}{1 \times 1}$$

$$\frac{1 \times 3 \quad 3 \times 1}{1 \times 1}$$

$$\frac{1 \times 3 \quad 3 \times 1}{1 \times 1}$$

Also, approximation in matrix form:

displacements  $u^e = H^e U^e$ ,  $U^e$ : vector of nodal displacements  
 $3 \times 1$

What is the dimension of  $U^e$ ? And  $H^e$ ?

$$(U^e)^T = \{ U_1^1 U_2^1 U_3^1 \quad U_1^2 U_2^2 U_3^2 \dots U_1^n U_2^n U_3^n \}$$

strains  $\epsilon^e = B^e U^e$

$$6 \times 1 \quad 6 \times 3n \quad 3n \times 1$$

Obviously  $H^e$  is obtained from  $\phi_k$  and  $B^e$  from their derivatives.

Replace in potential:

$$\begin{aligned} \Pi^e &= \int_{V^e} \frac{1}{2} \overbrace{U^T B^T}^{\epsilon^T} C \overbrace{B U}^{\epsilon} - \int_{V^e} U^T H^T f \, dV - \int_{S^e} U^T H^T t \, dS \\ &= \frac{1}{2} U^T \int_{V^e} B^T C B \, dV U - U^T \int_{V^e} H^T f \, dV - U^T \int_{S^e} H^T t \, dS \\ &\quad \underbrace{\hspace{10em}}_{K^e} \quad \underbrace{\hspace{10em}}_{U^T R^e} \end{aligned}$$

$$\Pi^e = \frac{1}{2} U^T K^e U - U^T R^e$$

where

$$K^e = \int_{V^e} B^T C B \, dV$$

$$\underbrace{3n \times 6 \quad 6 \times 6 \quad 6 \times 3n}_{3n \times 3n}$$

$$R^e = \int_{V^e} H^T f \, dV + \int_{S^e} H^T t \, dS$$

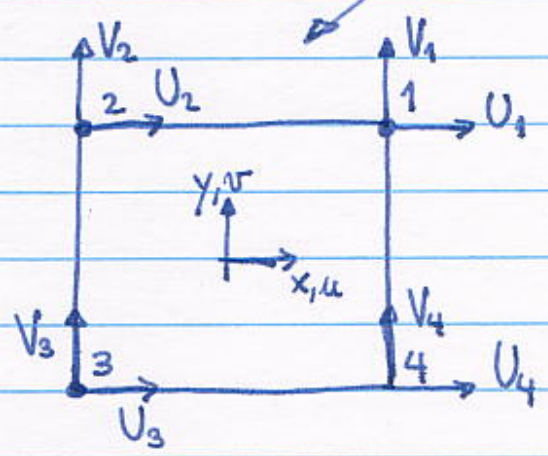
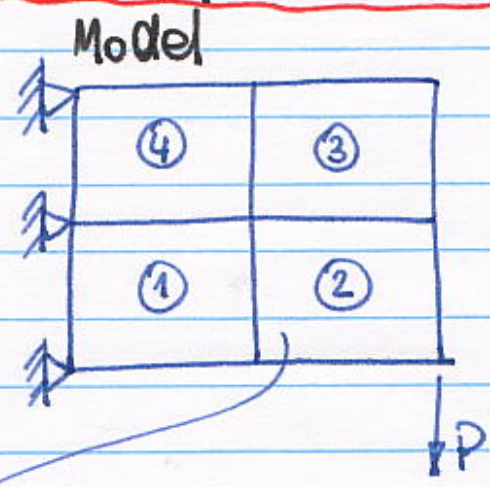
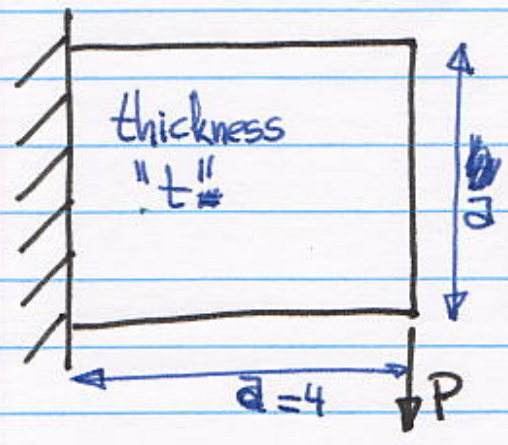
$$\underbrace{3n \times 3 \quad 3 \times 1}_{3n \times 1}$$

Upon minimization of  $\Pi^e$  with respect to nodal displacements  $U^e$  —

$$K^e U^e = R^e$$

$$3n \times 3n \quad 3n \times 1 \quad 3n \times 1$$

• Example: plane stress. linear square element



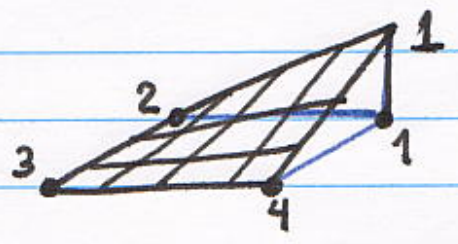
- 1  $\equiv$  (1, 1)
- 2  $\equiv$  (-1, 1)
- 3  $\equiv$  (-1, -1)
- 4  $\equiv$  (1, -1)

Basis functions:

$$\phi_1 = \frac{1}{4} (1+x)(1+y)$$

$$\phi_2 = \frac{1}{4} (1-x)(1+y)$$

$$\phi_3 = \frac{1}{4} (1-x)(1-y)$$



etc.

$$\phi_4 = \frac{1}{4} (1+x)(1-y)$$

$$u = \begin{Bmatrix} u_x \\ v_x \end{Bmatrix} = \begin{bmatrix} \phi_1 & 0 & \phi_2 & 0 & \phi_3 & 0 & \phi_4 & 0 \\ 0 & \phi_1 & 0 & \phi_2 & 0 & \phi_3 & 0 & \phi_4 \end{bmatrix} \begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \\ U_4 \\ V_4 \end{Bmatrix}$$

H

$$\epsilon = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} u_x \\ v_y \\ u_y + v_x \end{Bmatrix} = B U$$

$$B = \begin{bmatrix} \phi_{1,x} & 0 & \phi_{2,x} & 0 & \phi_{3,x} & 0 & \phi_{4,x} & 0 \\ 0 & \phi_{1,y} & 0 & \phi_{2,y} & 0 & \phi_{3,y} & 0 & \phi_{4,y} \\ \phi_{1,y} & \phi_{1,x} & \phi_{2,y} & \phi_{2,x} & \phi_{3,y} & \phi_{3,x} & \phi_{4,y} & \phi_{4,x} \end{bmatrix}$$

$$\phi_{1,x} = \frac{1}{4} (1+y)$$

$$\phi_{1,y} = \frac{1}{4} (1+x)$$

$$\phi_{2,x} = -\frac{1}{4} (1+y)$$

$$\phi_{2,y} = \frac{1}{4} (1-x)$$

$$\phi_{3,x} = -\frac{1}{4} (1-y)$$

$$\phi_{3,y} = -\frac{1}{4} (1-x)$$

$$\phi_{4,x} = \frac{1}{4} (1-y)$$

$$\phi_{4,y} = -\frac{1}{4} (1+x)$$

Replace in B, then in K with

$$C = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

See Mathematica file.