The finite element method II

In FEM I we derived basis functions of arbitrary order for the rod model:

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right)+f(x)=0$$

u specified at boundary or
$AE\frac{du}{dx}$ " " "

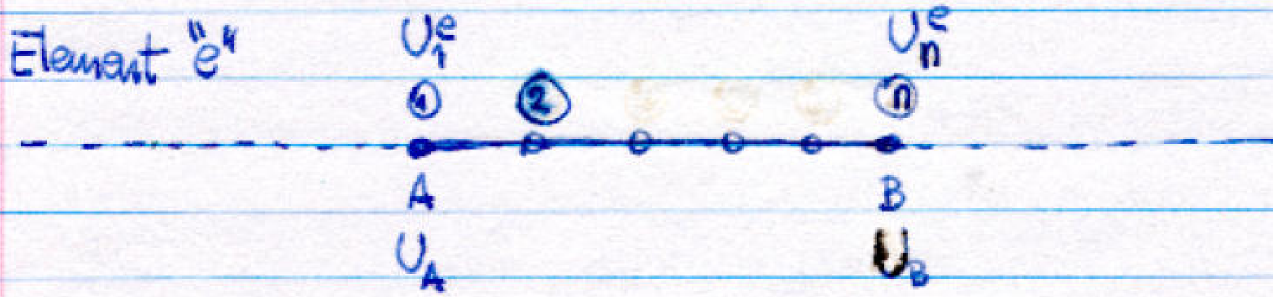
$\Longrightarrow$ approximation inside element "e"

$$\boxed{u_e(x)=\sum_{i=1}^{n}\phi_i^e(x)\,U_i^e} \qquad x_A<x<x_B$$

Properties of $\phi_i^e(x)$

- $\boxed{\phi_i^e(x_j)=\delta_{ij}}$ $\quad j=1,\dots,n$
- $\boxed{\sum_{i=1}^{n}\phi_i(x)=1}$ $\quad \forall x \quad x_A\leq x\leq x_b$

Today: use this approximation to solve Ritz approximation within element "e"



"Element boundary conditions":

$$u(x_1^e) = U_1^e \qquad u(x_n^e) = U_n^e$$

$$AE \left.\frac{du}{dx}\right|_{x=x_1^e} = P_1^e \qquad AE \left.\frac{du}{dx}\right|_{x=x_n^e} = P_n^e$$

PVD (alternatively PMPE)

$$\int_{x_1^e}^{x_n^e}\int_A \sigma \, \delta\varepsilon \, dA \, dx = \int_{x_1^e}^{x_n^e} q(x)\, \delta u \, dx + \sum P_k^e \, \delta u(x_k^e)$$

$$\int_{x_1^e}^{x_n^e} EA \frac{du}{dx} \, \delta \frac{du}{dx} \, dx = \int_{x_1^e}^{x_n^e} q(x)\, \delta u \, dx + \sum P_k^e \, \delta u(x_k^e)$$

$$\int_{x_1^e}^{x_n^e} EA \frac{du}{dx} \, \frac{d}{dx}\delta u \, dx = \int_{x_1^e}^{x_n^e} q(x)\, \delta u \, dx + \sum P_k^e \, \delta u(x_k^e)$$

Replace approximation inside element:

$$u_e = \sum_{i=1}^{n} \phi_i^e(x)\, U_i^e \qquad \delta u_e = \sum_{i=1}^{n} \phi_i^e(x)\, \delta U_i^e$$

$$\frac{du_e}{dx} = \sum_{i=1}^{n} \frac{d\phi_i^e(x)}{dx}\, U_i^e \qquad \frac{d\,\delta u_e}{dx} = \sum_{i=1}^{n} \frac{d\phi_i^e(x)}{dx}\, \delta U_i^e$$

PVD

$$\delta W_I^e = \int_{x_1^e}^{x_n^e} EA \left(\sum_{i=1}^{n} \frac{d\phi_i^e}{dx} U_i^e\right)\left(\sum_{j=1}^{n} \frac{d\phi_j^e}{dx}\, \delta U_j^e\right) dx$$

$$= \int \delta U_j^e \underbrace{\left[\int_{x_1^e}^{x_n^e} EA \frac{d\phi_j^e}{dx} \frac{d\phi_i^e}{dx}\, dx\right]}_{K_{ji}^e} U_i^e$$

$$\delta W_E^e = \int_{x_1^e}^{x_n^e} q(x) \sum_{i=1}^{n} \phi_i^e\, \delta U_i^e\, dx + \sum_{i=1}^{n} P_i^e \underbrace{\phi_{(i)}^e(x_i)}_{1} \delta U_i^e$$

$$= \delta U_j^e \underbrace{\left[\int_{x_1^e}^{x_n^e} q(x)\, \phi_j^e\, dx + \sum P_j^e\right]}_{R_j^e}$$

$$\delta W_I = \delta W_E$$

$$\delta U_j^e \; K_{ji}^e \, U_i^e = \delta U_j^e \; R_j^e$$

$$\delta U_j^e \left[K_{ji}^e U_i^e - R_j^e \right] = 0 \qquad \forall \, \delta U_j^e \text{ (admissible)}$$

$$\Leftrightarrow \qquad \boxed{K_{ji}^e \; U_i^e - R_j^e = 0}$$

Stiffness matrix:

$$\boxed{K_{ij}^e = \int_{x_1^e}^{x_n^e} AE \, \frac{d\phi_i^e}{dx} \frac{d\phi_j^e}{dx} \, dx}$$

- K can be computed given element type ("n") and A, E inside element.
- $K \in \mathbb{R}^{n \times n}$
- K symmetric
- K_{ij}^e can be interpreted as the force needed on node "j" when a unit displacement is applied at node "i".

- K_{ij} is singular why?

Force vector

$$\boxed{R_i^e = \int_{x_1^e}^{x_n^e} f^e(x)\, \phi_i^e \, dx + P_i^e}$$