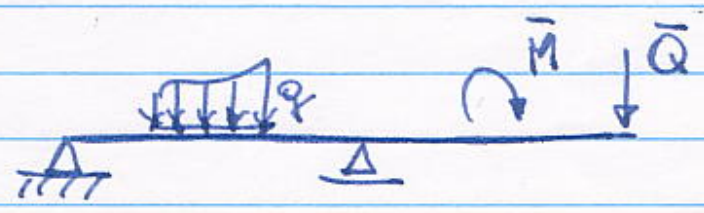


Finite element model of a beam (Euler-Bernoulli)

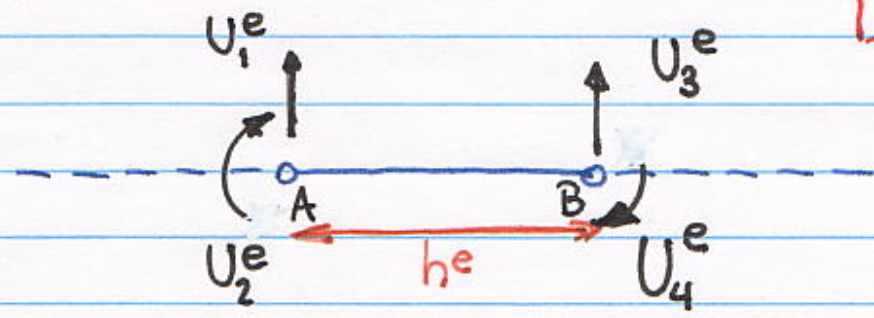
Governing equations:

$$\left\{ \begin{array}{l} (EI w''')' - q = 0 \quad 0 < x < L \\ \left. \begin{array}{l} w = \bar{w} \\ w' = \bar{w}' \end{array} \right\} \text{ on } S_u \quad (\text{displacement bc}) \\ \left. \begin{array}{l} EI w'' = \bar{M} \\ (EI w''')' = \bar{Q} \end{array} \right\} \text{ on } S_t \quad (\text{natural or traction bc}) \end{array} \right.$$



Approximation inside element

$$w_0^e(x) \approx \sum_{i=1}^n \phi_i^e U_i^e$$



"Element boundary conditions"

• displacement $w(x_1^e) = U_1^e$ $w'(x_1^e) = U_2^e$ $w(x_2^e) = U_3^e$ $w'(x_2^e) = U_4^e$

• natural

$$(EI w''')' \Big|_{x_A^e} = Q_1^e \quad (EI w''')' \Big|_{x_B^e} = -Q_3^e$$

$$EI w'' \Big|_{x_A^e} = Q_2^e \quad EI w'' \Big|_{x_B^e} = -Q_4^e$$

What do the first and second row represent?

Potential energy for the beam element

$$\Pi^e(w_0^e) = \int_{x_A}^{x_B} \left[E_e I_e \left(\frac{d^2 w_0^e}{dx^2} \right)^2 + w_0^e q_0^e \right] dx -$$

$$- P_1^e U_1^e - P_2^e U_2^e - P_3^e U_3^e - P_4^e U_4^e$$

What do the last four terms represent?

Derivation of basis function ϕ_i

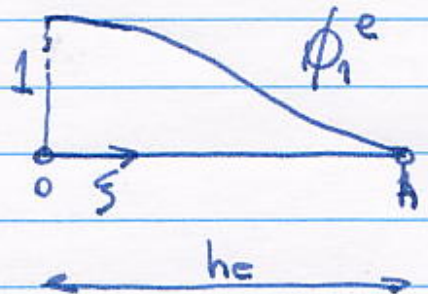
Need twice-differentiable, continuous, continuous-slope functions. The minimum polynomial order should be "three", so that non-zero shears are obtained at the nodes. The cubic polynomial also gives us four parameters to fit the four essential

boundary conditions at the nodes.

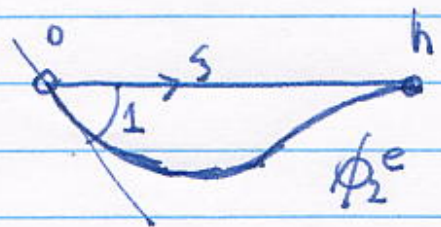
The resulting basis functions are the

Hermite cubic polynomials

$$\phi_1^e = 1 - 3\left(\frac{\xi}{h_e}\right)^2 + 2\left(\frac{\xi}{h_e}\right)^3$$



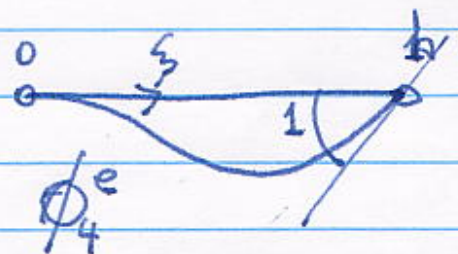
$$\phi_2^e = -\xi \left[1 - \left(\frac{\xi}{h_e}\right)^2 \right]$$



$$\phi_3^e = 3\left(\frac{\xi}{h_e}\right)^2 - 2\left(\frac{\xi}{h_e}\right)^3$$



$$\phi_4^e = -\xi \left[\left(\frac{\xi}{h_e}\right)^2 - \frac{\xi}{h_e} \right]$$



Finite element equations:

Replace approximation $w_0^e = \sum_{i=1}^4 \phi_i^e U_i^e$ into the

potential Π :

$$\Pi^e(U_i^e) = \int_{x_A}^{x_B} \left[\frac{E_e I_e}{2} \left(\sum_{j=1}^4 U_j^e \frac{d^2 \phi_j^e}{dx^2} \right)^2 + \sum_{i=1}^4 U_i^e \phi_i^e q_0^e \right] dx - \sum_{i=1}^4 P_i^e U_i^e$$

Apply PMPE: $\delta \Pi^e = 0 \iff \frac{\partial \Pi^e}{\partial U_i^e} = 0$

$$\frac{\partial \Pi^e}{\partial U_i^e} = \int_{x_A}^{x_B} \left[E_e I_e U_j^e \frac{d^2 \phi_j^e}{dx^2} \underbrace{\frac{\partial U_k^e}{\partial U_i^e}}_{\delta_{ik}} \frac{d^2 \phi_k^e}{dx^2} + q_0^e \underbrace{\frac{\partial U_k^e}{\partial U_i^e}}_{\delta_{ik}} \phi_k^e \right] dx - P_k^e \underbrace{\frac{\partial U_k^e}{\partial U_i^e}}_{\delta_{ik}} = 0$$

$$0 = \underbrace{\int_{x_A}^{x_B} E_e I_e \frac{d^2 \phi_i^e}{dx^2} \frac{d^2 \phi_j^e}{dx^2} dx}_{K_{ij}^e} U_j^e + \underbrace{\int_{x_A}^{x_B} q_0^e \phi_i^e dx}_{R_i^e} - P_i^e$$

\rightarrow K_{ij}^e element stiffness matrix
 $\rightarrow U_j^e = R_i^e$ element nodal displacements element force vector

For the case in which E_e and I_e are constant inside the element, these reduce to:

$$K^e = \frac{2EI}{h^3} \begin{bmatrix} 6 & -3h & -6 & -3h \\ & 2h^2 & 3h & h^2 \\ \text{sym} & & 6 & 3h \\ & & & 2h^2 \end{bmatrix}$$

$$R^e = \frac{-q_0 h}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix} + \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix}$$