

Signals and Systems

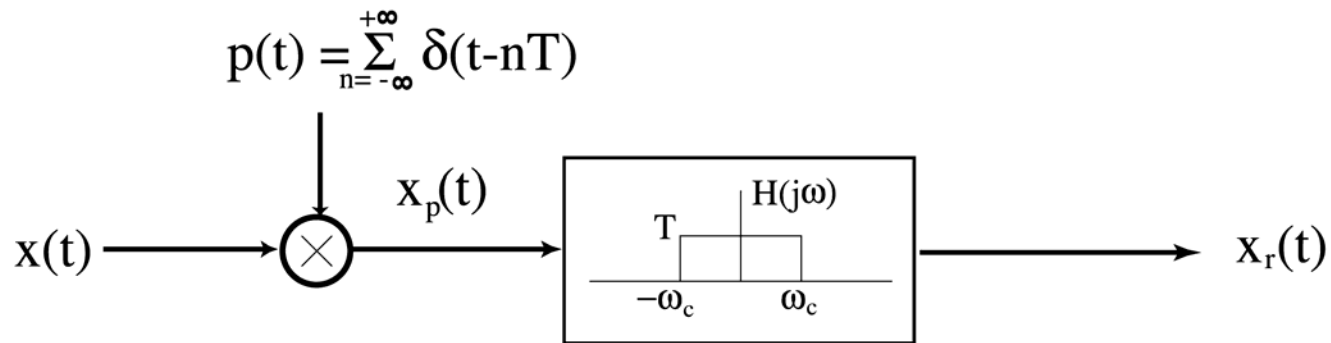
Fall 2003

Lecture #14

23 October 2003

1. Review/Examples of Sampling/Aliasing
2. DT Processing of CT Signals

Sampling Review

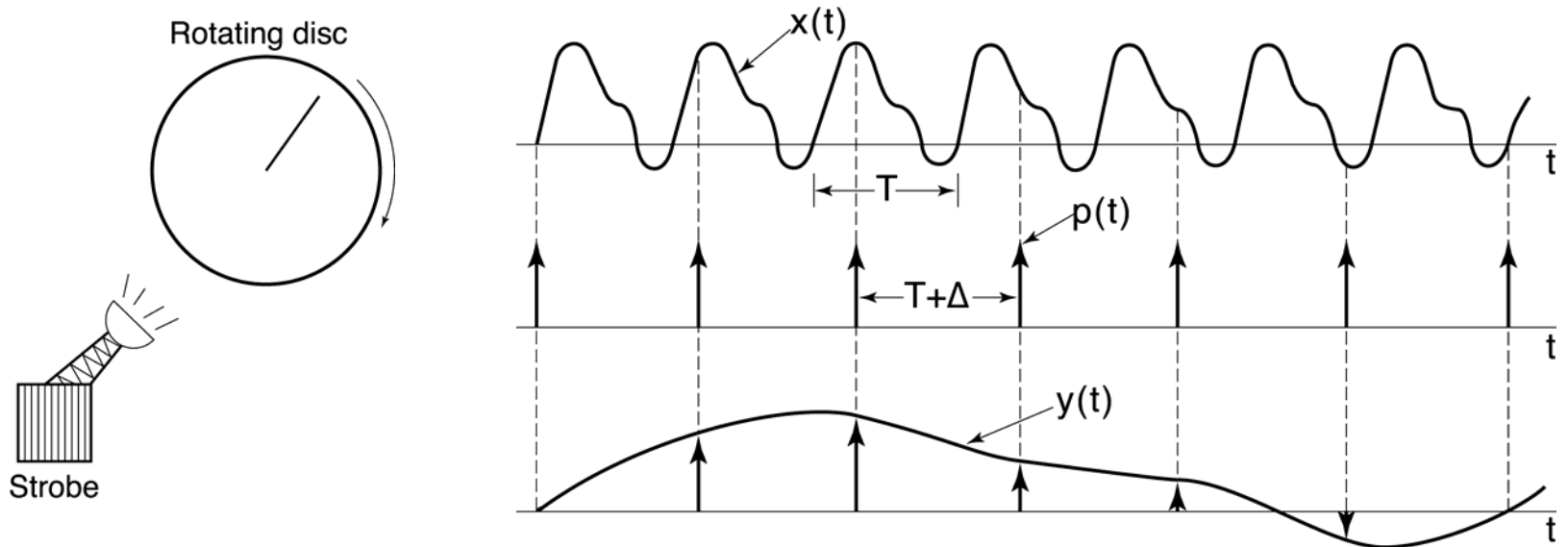


If $X(j\omega) = 0$, $|\omega| > \omega_M$ and $\omega_s = \frac{2\pi}{T} > 2\omega_M$
then, assuming we choose $\omega_M < \omega_c < \omega_s - \omega_M$:

$$x_r(t) = x(t)$$

Demo: Effect of aliasing on music.

Strobe Demo



$\Delta > 0$, strobed image moves forward, but at a slower pace

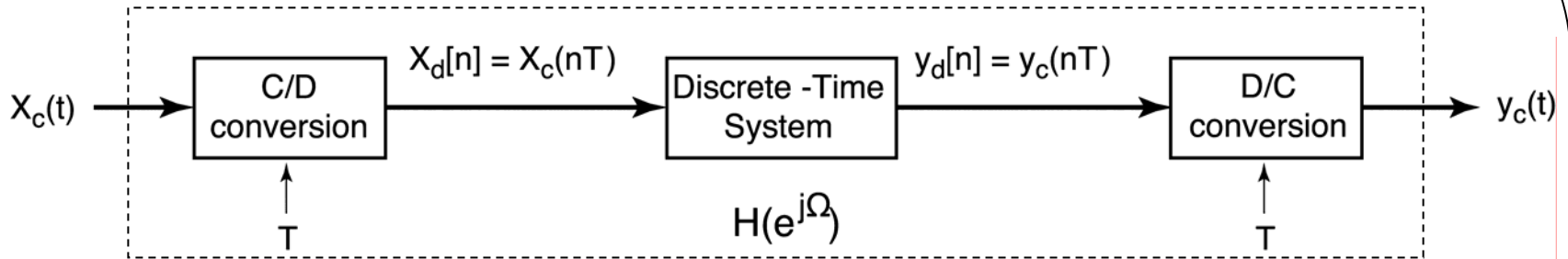
$\Delta = 0$, strobed image still

$\Delta < 0$, strobed image moves *backward*.

Applications of the strobe effect (*aliasing* can be useful sometimes):

— E.g., Sampling oscilloscope

DT Processing of *Band-Limited* CT Signals



Why do this?

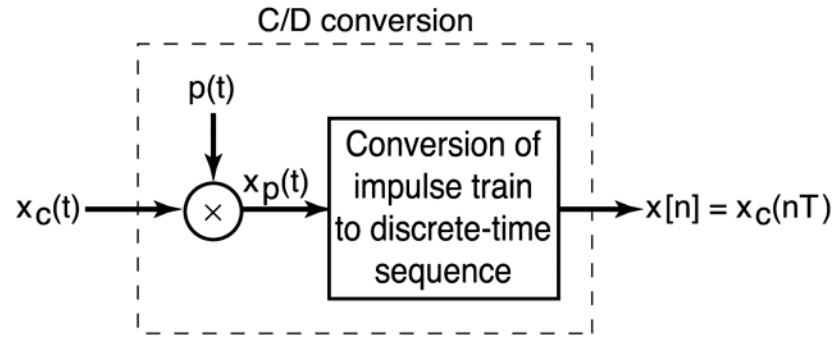
- Inexpensive, versatile, and higher noise margin.

How do we analyze this system?

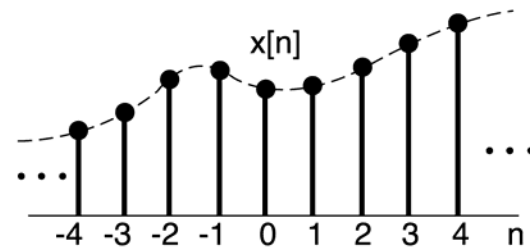
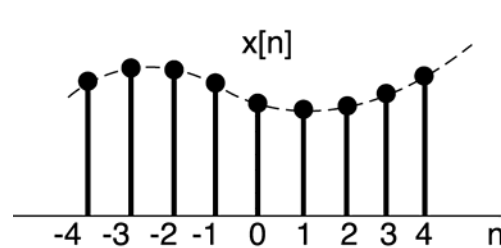
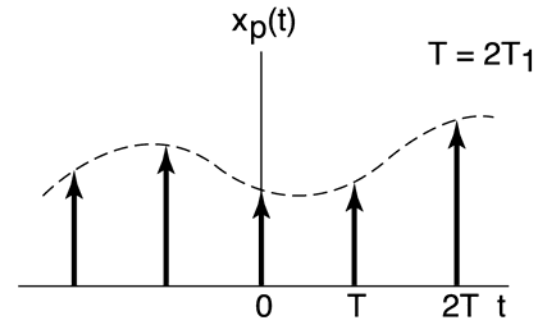
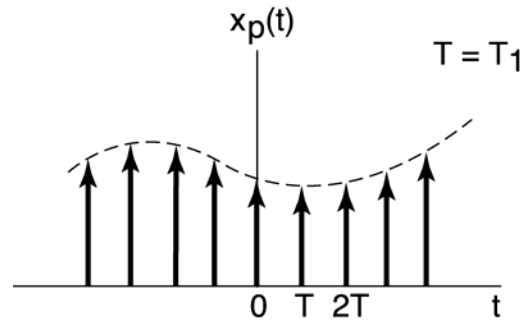
- We will need to do it in the frequency domain in both CT *and* DT
- In order to avoid confusion about notations, specify
 - ω — CT frequency variable
 - Ω — DT frequency variable ($\Omega = \omega T$)

Step 1: Find the relation between $x_c(t)$ and $x_d[n]$, or $X_c(j\omega)$ and $X_d(e^{j\Omega})$

Time-Domain Interpretation of C/D Conversion



Note: Not full analog/digital (A/D) conversion – not quantizing the $x[n]$ values



Frequency-Domain Interpretation of C/D Conversion

$$x_p(t) = x_c(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x_c(nT) \delta(t - nT)$$

$\Downarrow \mathcal{F}$

$$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\omega - k\omega_s)) = \sum_{n=-\infty}^{\infty} x_c(nT) e^{-j\omega nT} \quad (1)$$

CT — (periodic with period $\omega_s = 2\pi/T$)

$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_d[n] e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x_c(nT) e^{-j\Omega n} \quad (2)$$

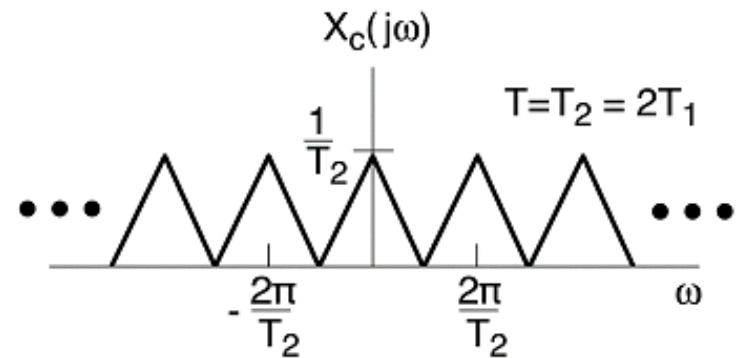
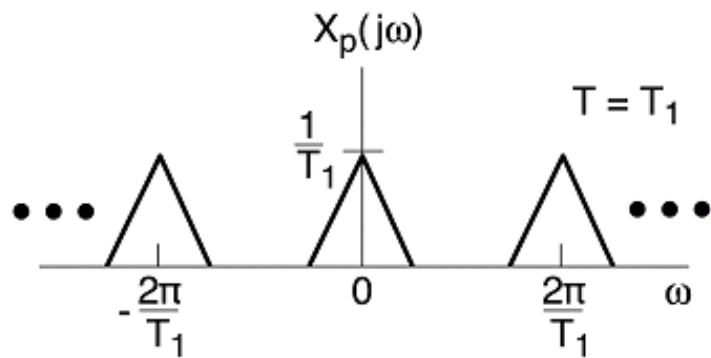
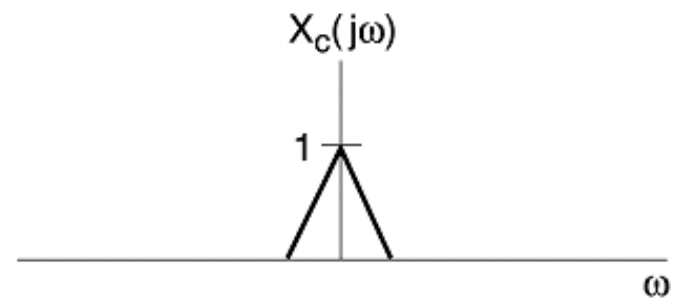
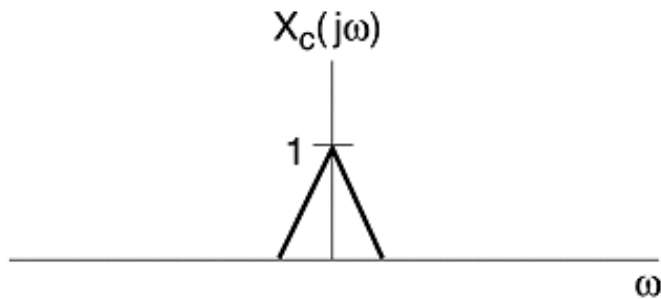
DT — (periodic with period 2π)

\Downarrow Compare Eqs. (1) & (2) and note $\Omega = \omega T$

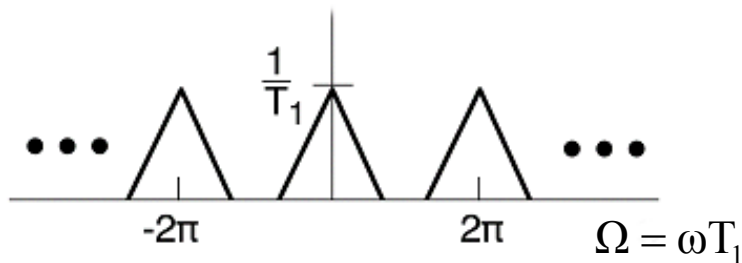
$$X_d(e^{j\Omega}) = X_p \left(j \left(\frac{\Omega}{T} \right) \right)$$

Note: $\omega_s \Leftrightarrow 2\pi$
CT DT

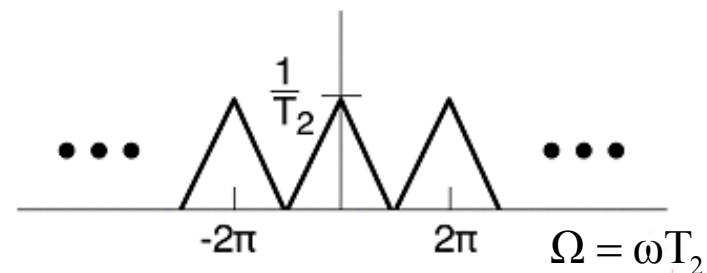
Illustration of C/D Conversion in the Frequency-Domain



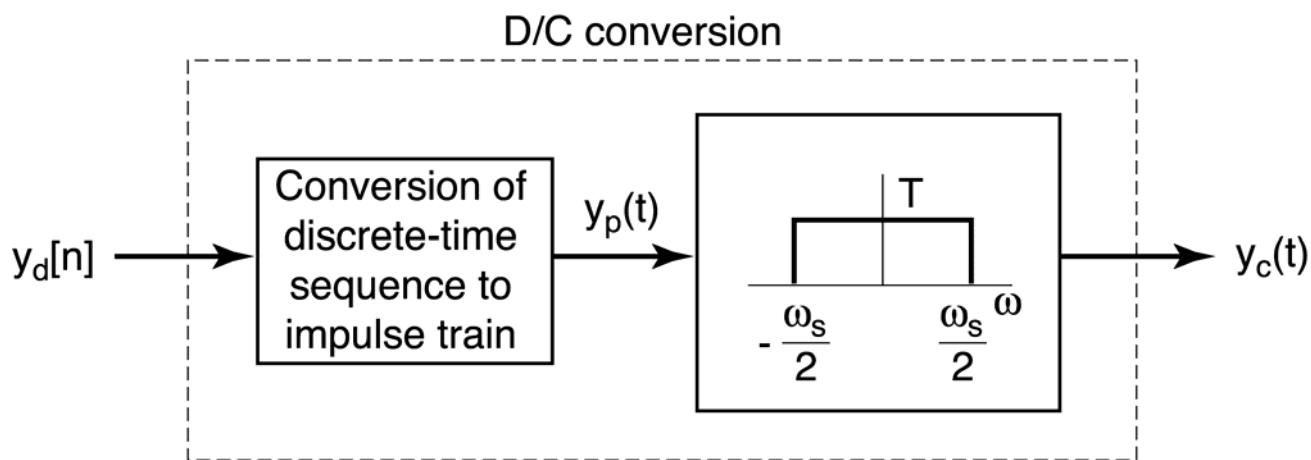
$$X_d(e^{j\Omega})$$



$$X_d(e^{j\Omega})$$



D/C Conversion $y_d[n] \rightarrow y_c(t)$
 Reverse of the process of C/D conversion

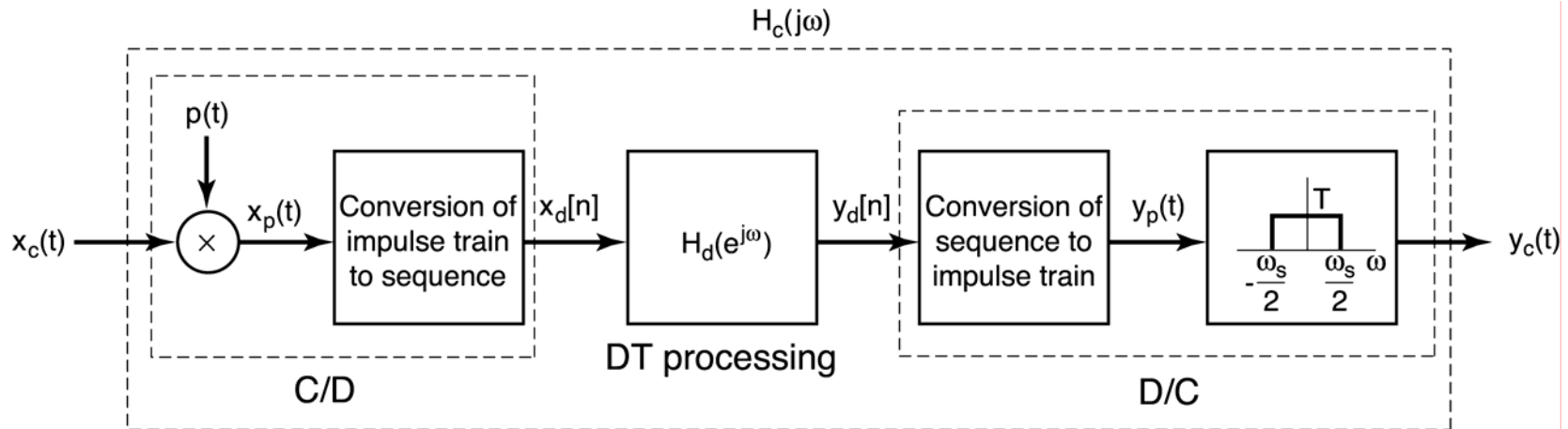


Again, $\Omega = \omega T$

$Y_p(j\omega) = Y_d(e^{j\omega T})$ — Reverses frequency scaling

$$Y_c(j\omega) = \begin{cases} TY_d(e^{j\omega T}), & |\omega| < \frac{\omega_s}{2} \text{ — bandlimited} \\ 0, & \text{otherwise} \end{cases}$$

Now the whole picture

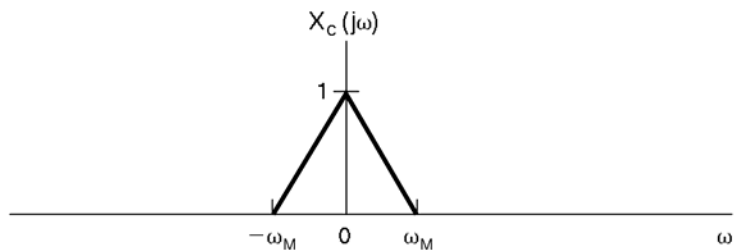


- Overall system is time-varying if sampling theorem is *not* satisfied
- It is LTI if the sampling theorem *is satisfied*, i.e. for bandlimited inputs $x_c(t)$, with

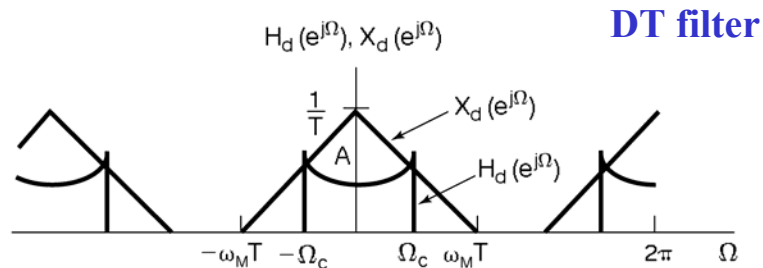
$$\omega_M < \frac{\omega_s}{2}$$
- When the input $x_c(t)$ is band-limited ($X(j\omega) = 0$ at $|\omega| > \omega_M$) and the sampling theorem is satisfied ($\omega_s > 2\omega_M$), then

$$Y_c(j\omega) = H_c(j\omega)X_c(j\omega) \longleftrightarrow y_c(t) = h_c(t) * x_c(t) \quad \text{LTI}$$

Frequency-Domain Illustration of DT Processing of CT Signals



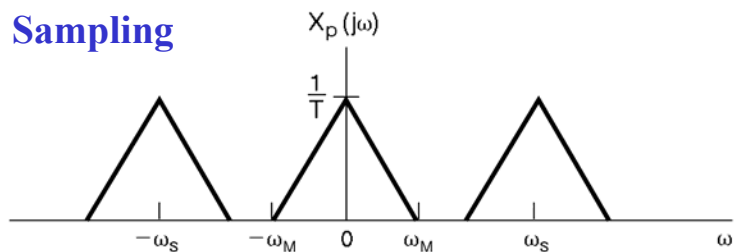
(a)



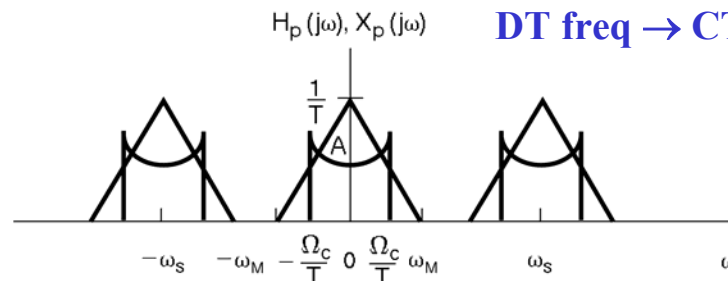
(d)

DT filter

Sampling



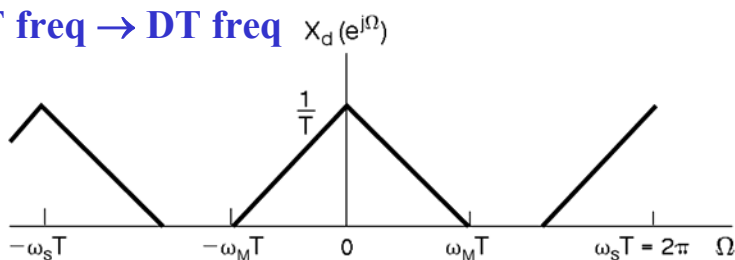
(b)



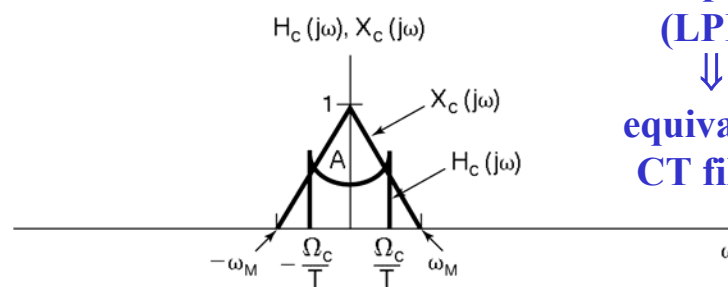
(e)

DT freq \rightarrow CT freq

CT freq \rightarrow DT freq



(c)



(f)

**Interpolate
(LPF)
 \downarrow
equivalent
CT filter**

Assuming No Aliasing

$$\begin{aligned} \text{Step 1 - C/D: } X_d(e^{j\Omega}) &= X_p(j\Omega/T) \quad - \text{ periodic} \\ \omega = \Omega/T &= \frac{1}{T} X_c(j\Omega/T), \quad -\pi < \Omega < \pi \text{ if no aliasing} \end{aligned}$$

$$\begin{aligned} \text{Step 2 - DT Filter: } Y_d(e^{j\Omega}) &= H_d(e^{j\Omega}) X_d(e^{j\Omega}) \\ &= \frac{1}{T} H_d(e^{j\Omega}) X_c(j\Omega/T), \quad -\pi < \Omega < \pi \end{aligned}$$

$$\begin{aligned} \text{Step 3 - D/C: } Y_c(j\omega) &= \begin{cases} T Y_d(e^{j\omega T}) = H_d(e^{j\omega T}) X_c(j\omega), & -\frac{\omega_s}{2} < \omega < \frac{\omega_s}{2} \\ 0, & \text{otherwise} \end{cases} \\ \Omega = \omega T & \end{aligned}$$

↓

$$H_c(j\omega) = \begin{cases} H_d(e^{j\omega T}), & |\omega| < \frac{\omega_s}{2} \\ 0, & \text{otherwise} \end{cases}$$

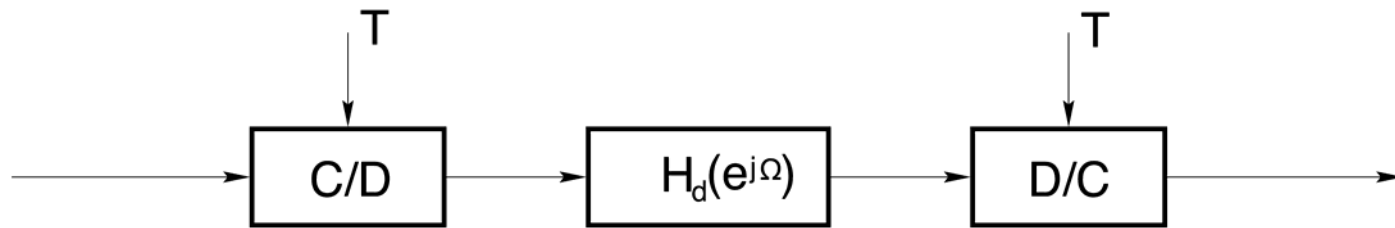
In practice, first specify the desired $H_c(j\omega)$, then design $H_d(e^{j\Omega})$.

Construction of Digital Differentiator

Bandlimited Differentiator

$$\text{Desired: } H_c(j\omega) = \begin{cases} j\omega, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

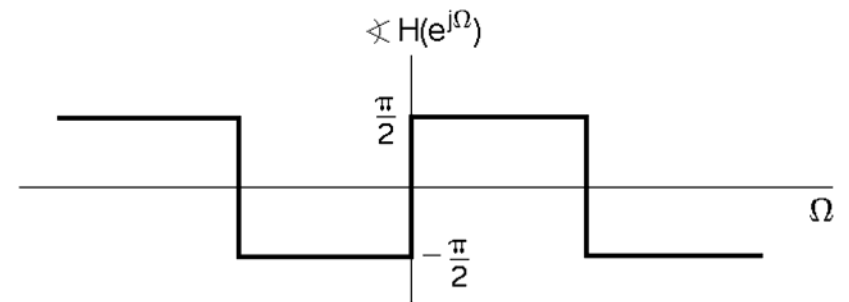
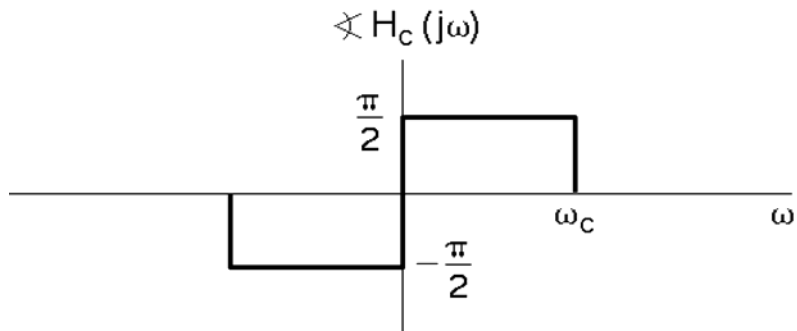
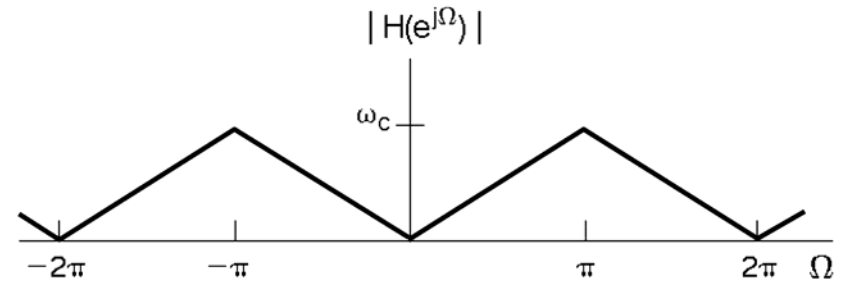
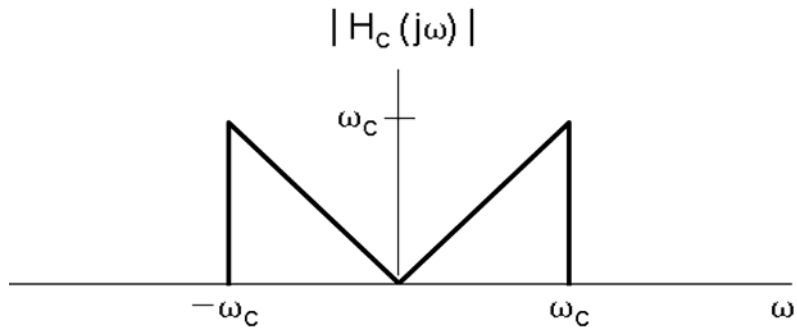
Set $\omega_s = 2\omega_c \Rightarrow T = \frac{2\pi}{\omega_s} = \frac{\pi}{\omega_c}$. Assume $\omega_M < \omega_c$ (Nyquist rate met)



Choice for $H_d(e^{j\Omega})$:

$$H_d(e^{j\Omega}) = \begin{cases} H_c(j\Omega/T), & |\Omega| < \pi \\ \text{periodic}, & |\Omega| \geq \pi \end{cases}$$
$$= j \left(\frac{\Omega}{T} \right) = j\omega_c \left(\frac{\Omega}{\pi} \right) \quad |\Omega| < \pi$$

Band-Limited Digital Differentiator (continued)



CT

DT