

# Señales y sistemas

## Otoño 2003

### Clase 5

18 de septiembre de 2003

1. Exponenciales complejos como funciones propias de sistemas LTI
2. Representación de las series de Fourier de señales periódicas de tiempo continuo (TC)
3. ¿Cómo se calculan los coeficientes de Fourier?
4. La convergencia y el fenómeno de Gibbs

## Retrato de Jean Baptiste Joseph Fourier

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Signals & Systems, 2<sup>a</sup> ed. Upper Saddle River, N.J.: Prentice Hall, 1997, pág. 179.









## What kinds of signals can we represent as “sums” of complex exponentials?

For Now: Focus on restricted sets of complex exponentials

CT:  $s = j\omega$  – purely imaginary,  
i.e., signals of the form  $e^{j\omega t}$

DT:  $z = e^{j\omega}$ ,  
i.e., signals of the form  $e^{j\omega n}$

Magnitude 1



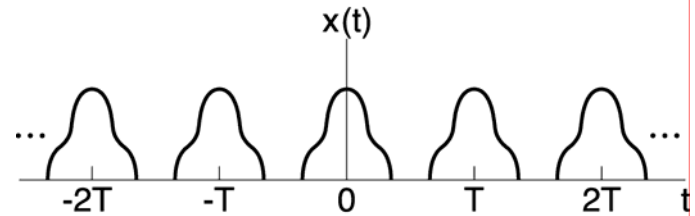
CT & DT Fourier Series and Transforms

↑  
Periodic Signals

# Fourier Series Representation of CT Periodic Signals

$$x(t) = x(t + T) \quad \text{for all } t$$

- smallest such  $T$  is the *fundamental period*
- $\omega_0 = \frac{2\pi}{T}$  is the *fundamental frequency*



$$e^{j\omega t} \text{ periodic with period } T \Leftrightarrow \omega = k\omega_0$$

↓

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi t/T}$$

- periodic with period  $T$
- $\{a_k\}$  are the *Fourier (series) coefficients*
- $k = 0$  DC
- $k = \pm 1$  first harmonic
- $k = \pm 2$  second harmonic

**Question #1:** How do we find the Fourier coefficients?

First, for simple periodic signals consisting of a few sinusoidal terms

$$\text{Ex: } x(t) = \cos 4\pi t + 2 \sin 8\pi t$$

Euler's relation  
(memorize!)  $= \frac{1}{2} [e^{j4\pi t} + e^{-j4\pi t}] + \frac{2}{2j} [e^{j8\pi t} - e^{-j8\pi t}]$

$$\omega_0 = 4\pi \quad T = \frac{2\pi}{\omega_0} = \frac{2\pi}{4\pi} = \frac{1}{2}$$

$$a_0 = 0 \text{ -- no dc component}$$

$$a_1 = \frac{1}{2}$$

$$a_{-1} = \frac{1}{2}$$

$$a_2 = \frac{1}{j}$$

$$a_{-2} = -\frac{1}{j}$$

$$a_3 = 0$$

$$a_{-3} = 0$$

⋮

- For *real* periodic signals, there are two other commonly used forms for CT Fourier series:

$$x(t) = a_0 + \sum_{k=1}^{\infty} [\alpha_k \cos k\omega_0 t + \beta_k \sin k\omega_0 t]$$

or

$$x(t) = a_0 + \sum_{k=1}^{\infty} [\gamma_k \cos(k\omega_0 t + \theta_k)]$$

- Because of the eigenfunction property of  $e^{j\omega t}$ , we will usually use the complex exponential form in 6.003.

- A consequence of this is that we need to include terms for *both* positive and negative frequencies:

$$e^{jk\omega_0 t} \quad , \quad e^{-jk\omega_0 t}$$

$$\text{Remember } \cos(k\omega_0 t) = \frac{1}{2}(e^{jk\omega_0 t} + e^{-jk\omega_0 t})$$

$$\text{and } \sin(k\omega_0 t) = \frac{1}{2j}(e^{jk\omega_0 t} - e^{-jk\omega_0 t})$$

## Now, the complete answer to Question #1

(Given  $x(t)$ ,  
how find  $a_k$ ?)

Suppose

- 1) multiply by  $e^{-jn\omega_0 t}$
- 2) integrate over one period

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$\Downarrow$

- 1) multiply by  $e^{-jn\omega_0 t}$
- 2) integrate over one period

$$\begin{aligned} \int_T x(t) e^{-jn\omega_0 t} dt &= \int_T \left( \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right) e^{-jn\omega_0 t} dt \\ &= \sum_{k=-\infty}^{\infty} a_k \left( \int_T e^{j(k-n)\omega_0 t} dt \right) \end{aligned}$$

(Here  $\int_T$  denotes integral over *any* interval of length  $T$  (one period).)

Next, note that

$$\begin{aligned} \int_T e^{j(k-n)\omega_0 t} dt &= \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases} \\ &= T\delta[k - n] \quad \text{Orthogonality} \\ &\Downarrow \end{aligned}$$

$$\int_T x(t)e^{-jn\omega_0 t} dt = \sum_{k=-\infty}^{\infty} a_k \left( \int_T e^{j(k-n)\omega_0 t} dt \right) = \sum_{k=-\infty}^{\infty} a_k \cdot T\delta[k-n]$$

$$\int_T x(t)e^{-jn\omega_0 t} dt = a_n T$$

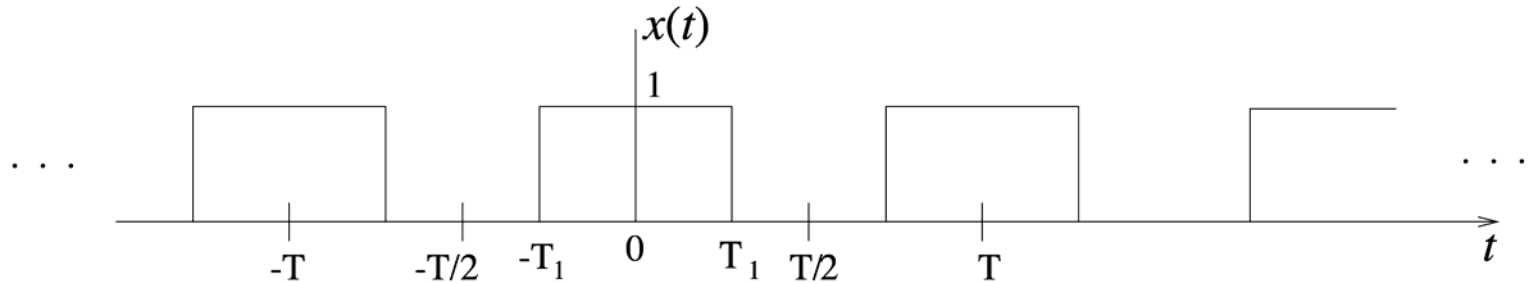


CT Fourier Series Pair ( $\omega_0 = \frac{2\pi}{T}$ )

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad (\text{Synthesis equation})$$

$$a_k = \frac{1}{T} \int_T x(t)e^{-jk\omega_0 t} dt \quad (\text{Analysis equation})$$

## Ex: Periodic Square Wave



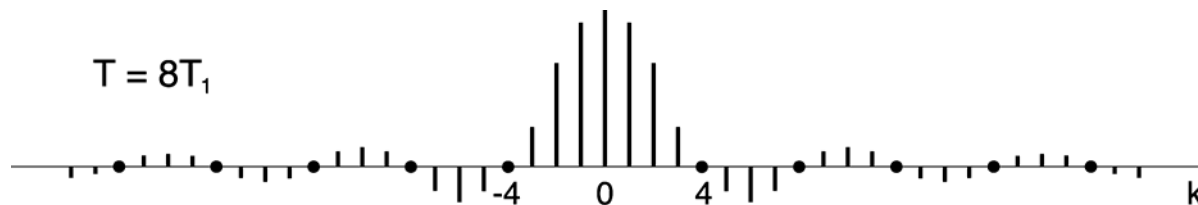
For  $k = 0$

$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} x(t) dt = \frac{2T_1}{T}$$

DC component  
is just the  
average

For  $k \neq 0$

$$\begin{aligned} a_k &= \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_{-T_1}^{T_1} e^{-jk\omega_0 t} dt \\ &= -\frac{1}{jk\omega_0 T} e^{-jk\omega_0 t} \Big|_{-T_1}^{T_1} = \frac{\sin k\omega_0 T_1}{k\pi} \quad \left( \omega_0 = \frac{2\pi}{T} \right) \end{aligned}$$



## Convergence of CT Fourier Series

- How can the Fourier series for the square wave possibly make sense?
- The key is: What do we *mean* by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad ?$$

- One useful notion for engineers: there is no *energy* in the difference

$$e(t) = x(t) - \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\int_T |e(t)|^2 dt = 0$$

(just need  $x(t)$  to have finite energy per period)

$$\int_T |x(t)|^2 dt < \infty$$

## Under a different, but reasonable set of conditions (the Dirichlet conditions)

**Condition 1.**  $x(t)$  is *absolutely integrable* over one period, i. e.

$$\int_T |x(t)| dt < \infty$$

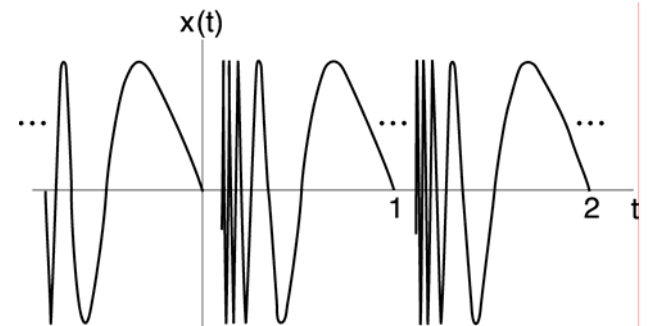
And

**Condition 2.** In a finite time interval,  $x(t)$  has a *finite* number of maxima and minima.

of

**Ex.** An example that violates Condition 2.

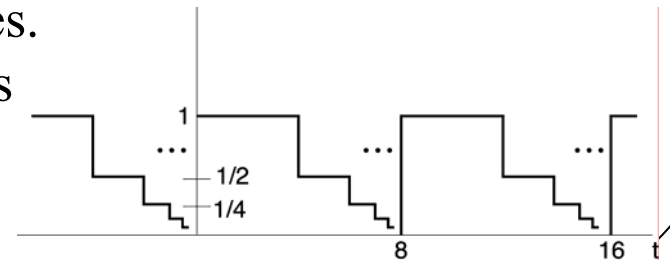
$$x(t) = \sin\left(\frac{2\pi}{t}\right) \quad 0 < t \leq 1$$



And

**Condition 3.** In a finite time interval,  $x(t)$  has only a *finite* number of discontinuities.

**Ex.** An example that violates Condition 3.



- Dirichlet conditions are met for the signals we will encounter in the real world. Then
  - The Fourier series =  $x(t)$  at points where  $x(t)$  is continuous
  - The Fourier series = “midpoint” at points of discontinuity
- Still, convergence has some interesting characteristics:

$$x_N(t) = \sum_{k=-N}^N a_k e^{jk\omega_0 t}$$

- As  $N \rightarrow \infty$ ,  $x_N(t)$  exhibits *Gibbs'* phenomenon at points of discontinuity

**Demo:** Fourier Series for CT square wave (Gibbs phenomenon).